Musterprüfung

Thema: Rechnen mit komplexen Zahlen

1.) Berechne die Lösungen von

a) $x^2+49=0$

b) $x^2-2x+5=0$

c) $x^2-10x+26=0$

d) $(x^2+9)\cdot(x^2-9)=0$

2.) Berechne die Summe und die Differenz von z_1 und z_2 ($z_1+z_2=?$ und $z_1-z_2=?$) für

a) $z_1=3i$ und $z_2=4+i$

b) $z_1=5-2i$ und $z_2=4+i$

c) $z_1=3-5i$ und $z_2=5-7i$

d) $z_1=a+bi$ und $z_2=ci$ $\{a,b,c\}\in\mathbb{R}$

e) $z_1=9-5i$ und $z_2=3+2i$

3.) Berechne (Wurzeln stehen lassen!)

a) $|3i|$

b) $|-5|$

c) $|-3+4i|$

d) $|7+3i|$

e) $|-5+3i|$

f) $|-8+15i|$

g) $|5-12i|$

h) $|-8+6i|$

4.) Berechne

a) $\mathrm{Re}(2-7i)$

b) $\mathrm{Im}(5+3i)$

c) $\mathrm{Im}(\overline{7-2i})$

d) $\overline{-4+5i}$

e) $|\overline{3-4i}|$

f) $\mathrm{Re}(7-3i)$

g) $\mathrm{Re}(11i)$

h) $3+5i+(\overline{3+5i})$

i) $\mathrm{Re}(2+3i)\cdot\mathrm{Im}(2+3i)$

j) $\mathrm{Re}((3+5i)-(2-3i))$

5.) Berechne

a) $(3-4i)\cdot(5+3i)$

b) $(1+3i)\cdot(2-5i)$

c) $(5-4i)\cdot(5+4i)$

d) $(i)^5$

e) $(3-2i)\cdot(3+3i)$

f) $(4-i)\cdot(3+2i)\cdot(5-2i)$

g) $(3+2i)^4$

6.) Berechne

a) $\frac{1}{2+i}$

b) $\frac{4-i}{3+2i}$

c) $\frac{5-3i}{4+i}$

d) $\frac{7-3i}{5+2i}$

e) $\frac{1}{(2-i)^2}$

f) $\frac{5-i}{2+3i}$

g) $\frac{6-i}{5+3i}$

h) $\frac{7-2i}{7+2i}$

7.) Die Grössen a und b seien reell. Berechne beide Grössen aus einer Gleichung wie folgt:

a) $2a+3i=b\cdot(2+i)+8$

b) $b\cdot(2a-5i)=4b-25i$

c) $3\cdot(a+ib)=5a-4+9i$

d) $(2a+5i)\cdot(b-i)=2ab+a+1+7i$

e) $a-2b+(b+3)i=6i-2$

8.) Berechne z aus

a) $\frac{1}{z}=3-i$

b) $(2+i)z=15i$

c) $2z+4iz=16i-2$

d) $5\frac{z+3i}{z-3i}=2+6i$

e) $17\cdot\frac{(2+i)z}{z+1}=5\cdot(5+3i)$

f) $(z+4i)\cdot(z-6)=z^2+16$

Musterlösungen

1a) $x^2 = -49 \rightarrow x = \pm\sqrt{-49} = \underline{\underline{\pm 7i}}$

b) $x^2 - 2x + 5 = 0$

a	b	c
1	-2	5

$D = b^2 - 4ac = 4 - 4 \cdot 1 \cdot 5$

$D = -16 < 0$

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{2 \pm \sqrt{-16}}{2 \cdot 1}$$

$$\sqrt{-16} = 4i \rightarrow x = \frac{2 \pm 4i}{2} = \underline{\underline{1 \pm 2i}} \quad \underline{\underline{x_1 = 1+2i}}, \underline{\underline{x_2 = 1-2i}}$$

c) $x^2 - 10x + 26 = 0$

a	b	c
1	-10	26

$D = b^2 - 4ac = 100 - 4 \cdot 1 \cdot 26$

$D = -4 < 0$

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

$$x = \frac{10 \pm \sqrt{-4}}{2 \cdot 1} = \frac{10 \pm 2i}{2} = \underline{\underline{5 \pm i}} \quad \underline{\underline{x_1 = 5+i}}, \underline{\underline{x_2 = 5-i}}$$

d) $(x^2+9) \cdot (x^2-9) = 0$

$$\left.\begin{array}{l} x^2 = 9 \rightarrow x = \pm 3 \\ x^2 = -9 \rightarrow x = \pm 3i \end{array}\right\} \underline{\underline{\mathbb{L} = \{3; -3; 3i; -3i\}}}$$

2a) $z_1 + z_2 = 3i + 4 + i = \underline{\underline{4+4i}}$, $z_1 - z_2 = 3i - 4 - i = \underline{\underline{-4+2i}}$

b) $z_1 + z_2 = 5 - 2i + 4 + i = \underline{\underline{9-i}}$, $z_1 - z_2 = 5 - 2i - 4 - i = \underline{\underline{1-3i}}$

c) $z_1 + z_2 = 3 - 5i + 5 - 7i = \underline{\underline{8-12i}}$, $z_1 - z_2 = 3 - 5i - 5 + 7i = \underline{\underline{-2+2i}}$

d) $z_1 + z_2 = a + bi + ci = \underline{\underline{a+(b+c)i}}$, $z_1 - z_2 = a + bi - ci = \underline{\underline{a+(b-c)i}}$

e) $z_1 + z_2 = 9 - 5i + 3 + 2i = \underline{\underline{12-3i}}$, $z_1 - z_2 = 9 - 5i - 3 - 2i = \underline{\underline{6-7i}}$

3a) $|3i| = \sqrt{0^2+3^2} = \underline{\underline{3}}$

b) $|-5| = \sqrt{5^2+0^2} = \underline{\underline{5}}$

c) $|-3+4i| = \sqrt{3^2+4^2} = \underline{\underline{5}}$

d) $|7+3i| = \sqrt{7^2+3^2} = \underline{\underline{\sqrt{58}}}$

e) $|-5+3i| = \sqrt{5^2+3^2} = \underline{\underline{\sqrt{34}}}$

f) $|-8+15i| = \sqrt{8^2+15^2} = \underline{\underline{17}}$

g) $|5-12i| = \sqrt{5^2+12^2} = \underline{\underline{13}}$

h) $|-8+6i| = \sqrt{8^2+6^2} = \underline{\underline{10}}$

4a) $Re(2-7i) = \underline{\underline{2}}$

b) $Im(5+3i) = \underline{\underline{3}}$

c) $Im(\overline{7-2i}) = Im(7+2i) = \underline{\underline{2}}$

d) $\overline{-4+5i} = \underline{\underline{-4-5i}}$

e) $|\overline{3-4i}| = |3+4i| = \sqrt{3^2+4^2} = \underline{\underline{5}}$

f) $Re(7-3i) = \underline{\underline{7}}$

g) $Re(11i) = \underline{\underline{0}}$

h) $3+5i+\overline{(3+5i)} = 3+5i+3-5i = \underline{\underline{6}}$

i) $Re(2+3i)\cdot Im(2+3i) = 2\cdot 3i = \underline{\underline{6i}}$

j) $Re((3+5i)-(2-3i)) = Re(1+7i) = \underline{\underline{1}}$

5a) $(3-4i)\cdot(5+3i) = 15+9i-20i+12 = \underline{\underline{27-11i}}$

b) $(1+3i)\cdot(2-5i) = 2-5i+6i+15 = \underline{\underline{17+i}}$

c) $(5-4i)(5+4i) = 25+20i-20i+16 = \underline{\underline{41}}$

d) $(i)^5 = (i)^2\cdot(i)^2\cdot i = (-1)\cdot(-1)\cdot i = \underline{\underline{i}}$

e) $(3-2i)\cdot(3+3i) = 9+9i-6i+6 = \underline{\underline{15+3i}}$

f) $(4-i)\cdot(3+2i)\cdot(5-2i) = (12+8i-3i+2)\cdot(5-2i) =$
$(14+5i)\cdot(5-2i) = 70-28i+25i+10 = \underline{\underline{80-3i}}$

g) $(3+2i)^4 = ((3+2i)^2)^2 = (9+6i+6i-4)^2 = (5+12i)^2$
$= 25+60i+60i-144 = \underline{\underline{-119+120i}}$

6a) $\frac{1}{2+i} = \frac{2-i}{(2+i)(2-i)} = \frac{2-i}{5} = \underline{\underline{\frac{2}{5}-\frac{1}{5}i}}$

b) $\frac{(4-i)(3-2i)}{3^2+2^2} = \frac{10-11i}{13} = \underline{\underline{\frac{10}{13}-\frac{11}{13}i}}$

c) $\frac{(5-3i)(4-i)}{17} = \frac{17-17i}{17} = \underline{\underline{1-i}}$

d) $\frac{(7-3i)(5-2i)}{29} = \frac{29-29i}{29} = \underline{\underline{1-i}}$

e) $\frac{1}{(2-i)^2} = \frac{1}{3-4i} = \frac{3+4i}{5} = \underline{\underline{\frac{3}{5}+\frac{4}{5}i}}$

f) $\frac{5-i}{2+3i} = \frac{(5-i)(2-3i)}{13} = \underline{\underline{\frac{7}{13}-\frac{17}{13}i}}$

g) $\frac{6-i}{5+3i} = \frac{27-23i}{34} = \frac{(6-i)(5-3i)}{34} = \underline{\underline{\frac{27}{34}-\frac{23}{34}i}}$

h) $\frac{7-2i}{7+2i} = \frac{(7-2i)(7-2i)}{53} = \frac{45-28i}{53} = \underline{\underline{\frac{45}{53}-\frac{28}{53}i}}$

7a) $2a+3i = 2b+bi+8 \rightarrow 3i = bi \rightarrow \underline{\underline{b=3}},\ a-b = 4 \rightarrow \underline{\underline{a=7}}$

b) $b\cdot(2a-5i)=4b-25i \rightarrow 2ab-5bi=4b-25i \rightarrow 5bi$
$=25i \rightarrow \underline{\underline{b=5}},\ 2ab=4b \longrightarrow \underline{\underline{a=2}}$

c) $3\cdot(a+ib)=5a-4+9i \rightarrow 3a+3bi=5a-4+9i \rightarrow$
$3bi=9i \rightarrow \underline{\underline{b=3}},\ 3a=5a-4 \rightarrow 2a=4 \rightarrow \underline{\underline{a=2}}$

d) $(2a+5i)\cdot(b-i)=2ab+a+1+7i \rightarrow$
$\cancel{2ab}-2ai+5bi+5=\cancel{2ab}+a+1+7i \rightarrow a+1=5 \rightarrow \underline{\underline{a=4}}$
$(5b-2a)i=7i \rightarrow 5b-8=7 \longrightarrow 5b=15 \longrightarrow \underline{\underline{b=3}}$

e) $a-2b+(b+3)i=6i-2 \rightarrow (b+3)i=6i \rightarrow \underline{\underline{b=3}},$
$a-2b=a-6=-2 \longrightarrow \underline{\underline{a=4}}$

8a) $\frac{1}{z}=3-i \longrightarrow z=\frac{1}{3-i}=\frac{3+i}{10}=\underline{\underline{\frac{3}{10}+\frac{1}{10}i}}$

b) $(2+i)z=15i \longrightarrow z=\frac{15i}{2+i}=\frac{15i(2-i)}{5}=3i(2-i)=\underline{\underline{3+6i}}$

c) $2z(1+2i)=2(8i-1) \longrightarrow z=\frac{8i-1}{1+2i}=\frac{(8i-1)\cdot(1-2i)}{5}=\underline{\underline{3+2i}}$

d) $5\cdot\frac{z+3i}{z-3i}=2+6i=2\cdot(1+3i) \longrightarrow 5z+15i=(2+6i)\cdot(z-3i)$
$5z+15i=2z-6i+6iz+18 \longrightarrow 3z-6iz=18-21i \rightarrow$
$z-2iz=(1-2i)z=6-7i \rightarrow z=\frac{6-7i}{1-2i}=\frac{(6-7i)(1+2i)}{5}$
$\underline{\underline{z=4+i}}$

e) $17\cdot\frac{(2+i)z}{z+1}=5\cdot(5+3i)=25+15i \rightarrow (34+17i)z=$
$(25+15i)\cdot(z+1) \longrightarrow (34+17i-25-15i)z=25+15i$
$(9+2i)z=5(5+3i)=\frac{5(5+3i)(9-2i)}{85}=\underline{\underline{3+i}}$

f) $(z+4i)\cdot(z-6)=z^2+16 \rightarrow \cancel{z^2}-6z+4zi-24i=\cancel{z^2}+16 \rightarrow$
$z(-6+4i)=16+24i=8(2+3i) \rightarrow z=\frac{4(2+3i)}{-3+2i}=$
$\frac{-4\cdot(2+3i)(3+2i)}{13}=\frac{-4\cdot 13i}{13}=\underline{\underline{-4i}}$

Rechnen mit komplexen Zahlen

Imaginäre Einheit: $i=\sqrt{-1}$, $i^2=-1$, $i^3=-i$, $i^4=1$

Reelle quadratische Gleichung mit konjugiert komplexen Zahlenpaar als Lösung:

$$ax^2+bx+c=0 \quad \text{mit} \quad b^2-4ac<0:$$

$$x=\frac{1}{2a}\left[-b\pm\sqrt{4ac-b^2}\,i\right]$$

Algebraische Form: $z=x+yi$, z.B. $z=3-2i$

x: Realteil , yi: Imaginärteil

Konjugiert komplexe Zahl: $z=x+iy \rightarrow \bar{z}=x-iy$

Die Funktionen $Re(z)$ und $Im(z)$:

Es gilt $Re(x+iy)=x$ und $Im(x+iy)=y$

Betrag: $|x+iy|=\sqrt{x^2+y^2}$, z.B. $|4-3i|=\sqrt{4^2+3^2}=5$

und es gilt $\bar{z}z=|z|^2$, z.B. $(4-3i)\cdot(4+3i)=4^2+3^2=5^2=25$

Addition: $(a+ib)+(c+id)=a+c+(b+d)i$

Subtraktion: $(a+ib)-(c+id)=a-c+(b-d)i$

Multiplikation: $(a+ib)\cdot(c+id)=ac-bd+(ad+bc)i$, z.B.

$(3-5i)\cdot(4+7i)=3\cdot4-5i\cdot7i+(3\cdot7-5\cdot4)i$

$=12-35\cdot(-1)+(21-20)i=47+i$

Division: $\dfrac{a+ib}{c+id}=\dfrac{(a+ib)(c-id)}{(c+id)(c-id)}=\dfrac{(a+ib)(c-id)}{c^2+d^2}$, z.B.

$\dfrac{3-7i}{1+2i}=\dfrac{(3-7i)(1-2i)}{1^2+2^2}=\dfrac{3\cdot1-7\cdot2+(3\cdot(-2)+(-7)\cdot1)i}{5}$

$=-2.2-2.6i$

Komplexe Gleichungen mit zwei reellen Lösungsvariablen.

Beispiel: $3a-8i=b+5-2ib$, wobei $\{a,b\}\in\mathbb{R}$.

$Re\rightarrow 3a=b+5\rightarrow a=(b+5)/3$

$Im\rightarrow -8i=-2ib\rightarrow b=4\rightarrow a=(b+5)3=3$

Komplexe lineare Gleichungen: Beispiel $3z+5i=12-2iz$

Zuerst z isolieren: $(3+2i)z=12-5i$ dann z ausrechnen

$z=(12-5i)/(3+2i)=2-3i$